

Breakdown of adiabatic transfer schemes in the presence of decay

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In atomic physics, adiabatic evolution is often used to achieve a robust and efficient population transfer. Many adiabatic schemes have also been implemented in optical waveguide structures. Recently there has been increasing interests in the influence of decay and absorption, and their engineering applications. Here it is shown that contrary to what is often assumed, even a small decay can significantly influence the dynamical behaviour of a system, above and beyond a mere change of the overall norm. In particular, a small decay can lead to a breakdown of adiabatic transfer schemes, even when both the spectrum and the eigenfunctions are only slightly modified. This is demonstrated for the decaying version of a STIRAP scheme that has recently been implemented in optical waveguide structures. It is found that the transfer property of the scheme breaks down at a sharp threshold, which can be estimated by simple analytical arguments.

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Introduction. The adiabatic theorem of Hermitian quantum mechanics provides the basis for many experimental schemes to manipulate and control quantum systems. Prominent examples from atomic physics include the RAP and STIRAP schemes [1] for population transfer in effective two or three level systems. In many applications, however, quantum systems are open, that is, their states have finite lifetimes described by complex energies. These more realistic situations cannot be described by Hermitian quantum mechanics, but require an additional anti-Hermitian part in the Hamiltonian (see, e.g., [2, 3] and references therein). Due to the analogy of the time dependent Schrödinger equation and the paraxial approximation for light propagation in optical media [4, 5], the dynamics generated by non-Hermitian Hamiltonians can conveniently be realised using optical systems [6].

It has previously been pointed out in the literature that the adiabatic theorem does not necessarily hold for non-Hermitian systems [7], and it has recently been shown that this can lead to new effects in the “deep” non-Hermitian regime in the presence of exceptional points [8]. However, it is often assumed that *small* decay rates do not modify the systems behaviour drastically, and their effects on adiabatic behaviour seem hitherto not to have been fully appreciated.

Here we show that even a small decay rate, which does not modify the static behaviour of a system significantly, can lead to a breakdown of adiabatic transfer properties. We demonstrate this fact for a STIRAP-related scheme, which is readily implemented in optical waveguide structures. In this scheme a sharp threshold for the breakdown of the transfer property is observed, which we calculate to a good approximation using simple analytical arguments.

Breakdown of adiabaticity in non-Hermitian systems. The adiabatic theorem states that a system initially prepared in an eigenstate remains in the corresponding instantaneous eigenstate if the system parameters are var-

ied infinitely slowly and the corresponding energy level is nondegenerate at all times. For finite parameter variations, there are small transition amplitudes between the adiabatic states, which are typically of order $O(\epsilon)$, where ϵ denotes the small adiabatic parameter [9, 10]. In the case of complex eigenvalues, however, the amplitudes of the instantaneous eigenstates are themselves exponentially decreasing in time. This can lead to a situation in which the small nonadiabatic transition amplitude grows exponentially in time relative to the adiabatic amplitude, if the adiabatic state is not the one with the smallest decay rate. In other words, the effect is caused by the dominance of the single gain mode of the time evolution operator [8]. We will now demonstrate how this general phenomenon can lead to a breakdown of the STIRAP scheme when the final state has a finite lifetime.

The STIRAP scheme. We begin with a brief summary of the conventional STIRAP scheme for a three-level quantum system described by a time dependent Hamiltonian of the form [1, 11]

$$H(t) = \begin{pmatrix} 0 & v(t) & 0 \\ v(t) & 0 & w(t) \\ 0 & w(t) & 0 \end{pmatrix}. \quad (1)$$

The instantaneous energies of the system are given by

$$E_0 = 0, \quad E_{\pm} = \pm \sqrt{v^2 + w^2}, \quad (2)$$

with the corresponding eigenstates

$$|\varphi^0\rangle = \begin{pmatrix} \cos \theta \\ 0 \\ -\sin \theta \end{pmatrix}, \quad |\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin \theta \\ \pm 1 \\ \cos \theta \end{pmatrix}, \quad (3)$$

where $\tan \theta = v/w$.

A variation of the coupling parameters $v(z)$ and $w(z)$ fulfilling $\frac{v(t_{\text{initial}})}{w(t_{\text{initial}})} \rightarrow \infty$ and $\frac{v(t_{\text{final}})}{w(t_{\text{final}})} \rightarrow 0$ rotates the

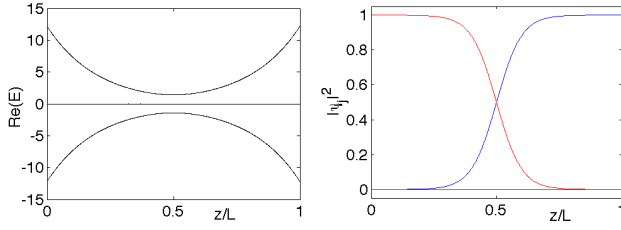


FIG. 1. The left figure shows the eigenvalues of the Hamiltonian (1), with the parameter dependence (4) for $a = 5$. The right figure shows the squared absolute values of the components of the adiabatic eigenstate $|\varphi^0\rangle$ (first, second, and third component shown in blue, green, and red, respectively).

instantaneous eigenstate $|\varphi^0\rangle$ from $|3\rangle$ to $|1\rangle$. Thus, adiabatic parameter variation leads to a complete population transfer from $|3\rangle$ to $|1\rangle$. Furthermore, since the second component of $|\varphi^0\rangle$ is identically zero at all times, the level $|2\rangle$ is never significantly populated during the process. The famous STIRAP scheme of atomic physics is an example of this adiabatic population transfer scheme [1]. In this context, while the states $|1\rangle$ and $|3\rangle$ are usually assumed stable, the intermediate state $|2\rangle$ generally couples to a continuum of other states, leading to a decay. Due to the vanishing population in $|2\rangle$, however, the scheme is insensitive to this decay [1]. Surprisingly, what appears not to have been noticed so far is that a decay from one of the other states can lead to a breakdown of the scheme. In what follows we shall demonstrate this for a decay from the final state, which is in practice often an excited energy state and thus less stable than the initial state.

Recently, STIRAP-type schemes have also been implemented in optical waveguides [12, 13], where the propagation distance z takes the role of time. The optical realisation allows for a straightforward experimental implementation of an additional decay of varying strength using absorbing materials. Thus, in what follows we shall proceed our analysis in the waveguide context. In particular, we study a generalisation of the setup investigated in [12], which consists of two parallel (left and right) waveguides with an additional diagonally directed central waveguide. Since the couplings between neighbouring waveguides depend exponentially on their distances, this system can be described by a Hamiltonian of the type (1) with couplings of the form

$$v(z) = 1/w(z) = e^{-a(z-L/2)/L}. \quad (4)$$

The instantaneous eigenvalues of this system are depicted in the left panel in Fig. 1. The right panel shows the components of the instantaneous eigenstate corresponding to the zero energy eigenvalue. For sufficiently large values of a and L , a complete intensity transfer between the right and the left waveguide is achieved without ever significantly populating the central waveguide, as has been experimentally demonstrated in [12].

STIRAP with decay in the final state. Let us now in-

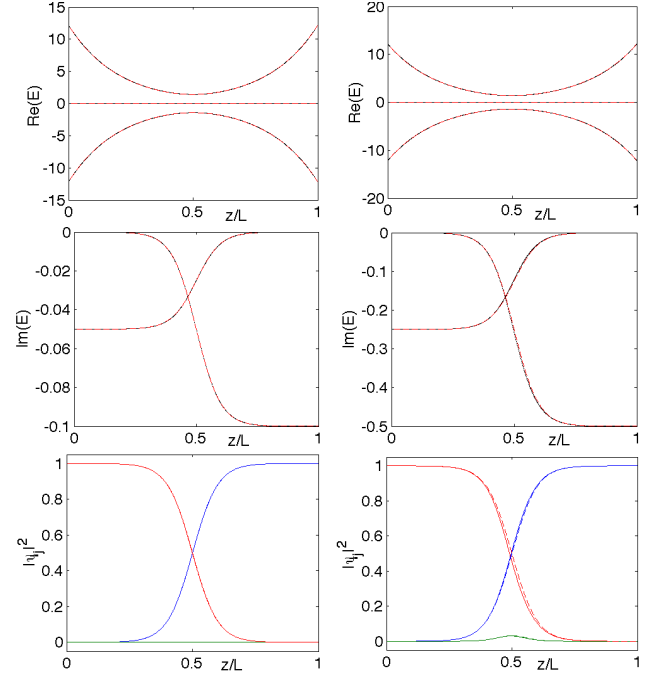


FIG. 2. Real (top) and imaginary (middle) parts of the energies for $\gamma = 0.1$ (left) and $\gamma = 0.5$ (right), and $a = 5$. The solid black line shows the numerically obtained eigenvalues, and the dashed red line those obtained from first order perturbation theory. The solid lines in the lower panel show the components of the adiabatic eigenstate $|\varphi^0\rangle$ (colours as in figure 1), the dashed lines show the perturbative result.

investigate how absorption in the target (left) waveguide modifies this transfer scheme. The additional decay is modelled by an imaginary energy $-i\gamma$ in the first diagonal element of the Hamiltonian (1). The eigenvalues for small γ can be obtained via first order perturbation theory that leaves the real parts unaltered and yields the additional imaginary parts:

$$\begin{aligned} \text{Im}(E_0) &\approx -\gamma \cos^2 \theta = -\gamma \frac{w^2}{v^2 + w^2}, \\ \text{Im}(E_{\pm}) &\approx -\gamma \frac{1}{2} \sin^2 \theta = -\gamma \frac{v^2}{2(v^2 + w^2)}. \end{aligned} \quad (5)$$

The eigenstate corresponding to E_0 , in first order perturbation theory, is given by

$$|\varphi^0\rangle = \begin{pmatrix} \cos \theta \\ i\gamma \cos \theta \sin \theta / \sqrt{v^2 + w^2} \\ -\sin \theta \end{pmatrix}. \quad (6)$$

In the upper and middle panels of Fig. 2 we show the numerically obtained energy levels in comparison to the first order perturbation theory for $a = 5$, and two different values of γ as a function of z/L . The energies are well described by the perturbative equation. In the lower panels of the figure the components of the instantaneous eigenstate $|\varphi^0\rangle$ are shown. As in the Hermitian case, this state populates the right waveguide for $z = 0$, while it

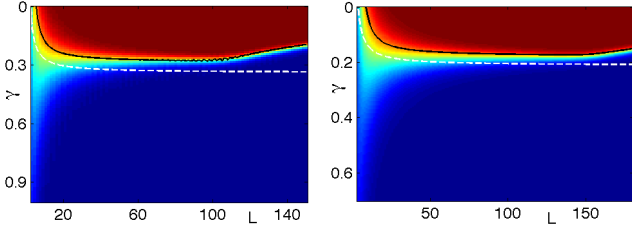


FIG. 3. Numerically obtained transfer probability ($P = 1$ is red, $P = 0$ is blue) as a function of L and γ for $a = 5$ (left) and $a = 8$ (right). The dashed white line shows the analytical transition boundary based on the amplification of the nonadiabatic coupling for the Hermitian system using the approximation (17), the solid black line is the semianalytical critical boundary using the numerically obtained nonadiabatic coupling for the non-Hermitian system with $\gamma = 0.2$.

is localised in the left for $z = L$. However, the central waveguide is also partly populated for intermediate values of z .

Assuming that the adiabatic theorem holds, one would conclude from the behaviour of the eigenvalues and eigenstates that the STIRAP scheme should lead to a full transfer of the population (up to an overall decay) for moderate values of γ . However, a numerical simulation of the total transferred probability, as a function of γ and L , yields a different result, as depicted in Fig. 3: There is a sharp transition for relatively small values of γ at which the STIRAP scheme breaks down. This is related to the relative growth of the nonadiabatic transitions mentioned above. In what follows we shall discuss in detail how this transition boundary emerges.

Adiabatic transition boundary. The transition probability to the left waveguide is defined by

$$P = \frac{|\langle 1 | \psi(L) \rangle|^2}{\sum_{n=1}^3 |\langle n | \psi(L) \rangle|^2}. \quad (7)$$

Here we consider the transition probability starting initially in the state $|\varphi^0\rangle$, corresponding approximately to the right waveguide, that is,

$$|\psi(0)\rangle = |\varphi^0(0)\rangle \approx -|3\rangle. \quad (8)$$

Let us express the solution of the time dependent Schrödinger equation as a sum of adiabatic and non-adiabatic parts in the instantaneous eigenbasis:

$$|\psi(z)\rangle = \psi_{ad}(z)|\varphi^0(z)\rangle + \sum_{j=\pm} \psi_{nonad}(z)|\varphi^j(z)\rangle, \quad (9)$$

where the nonadiabatic coefficient of the instantaneous states $|\varphi^\pm\rangle$ are equal due to the symmetry of the problem. On account of the relation $|\varphi^0(L)\rangle \approx |1\rangle$, the transfer probability in Eq. (7) can be estimated as

$$P = \frac{|\psi_{ad}(L)|^2}{|\psi_{ad}(L)|^2 + 2|\psi_{nonad}(L)|^2}. \quad (10)$$

In the Hermitian case the transfer is successful as long as the nonadiabatic term is negligible, which is the case for sufficiently large values of L . However, in the non-Hermitian case, we have to take into account that the adiabatic states have a finite lifetime, that is, they decay.

To a good approximation the transition between the instantaneous states occurs in the vicinity of the avoided crossing of the energies, close to $z = L/2$ (see Fig. 2). The states $|\varphi^\pm\rangle$ decay only slowly after this point, since the imaginary parts of their energies are small. Thus we find

$$|\psi_{nonad}(L)| \approx \sqrt{P_{nonad}}, \quad (11)$$

where P_{nonad} denotes the nonadiabatic transition probability due to the finite parameter variation. The state $|\varphi^0\rangle$, which we wish to follow adiabatically, on the other hand, has a small imaginary energy and is approximately stable before the nonadiabatic transition takes place, while the remaining population decays after the transition. Thus we estimate

$$|\psi_{ad}(L)| = \sqrt{1 - 2P_{nonad}} \exp\left(\int_0^L \text{Im } E_0 dz\right) \approx \sqrt{1 - 2P_{nonad}} e^{-\gamma L/2}, \quad (12)$$

where the integral is calculated using the eigenvalues in first order perturbation theory in equation (5). The factor of 2 in front of the nonadiabatic transition probability accounts for transitions into the two states $|\varphi^\pm\rangle$.

The population transfer is successful if

$$|\psi_{ad}(L)|^2 \gg 2|\psi_{nonad}(L)|^2. \quad (13)$$

Treating (13) as an equality and using (11) and (12), we obtain the threshold value

$$\gamma_{cr} = \ln\left(\frac{1}{2P_{nonad}} - 1\right)/L. \quad (14)$$

In the same way, the characteristic width of the threshold can be estimated by $\delta\gamma_{cr} \sim 2/L$. Since $\delta\gamma_{cr} \ll \gamma_{cr}$ for small P_{nonad} in Eq. (14), the observed threshold appears to be sharp.

Since we expect γ_{cr} to be relatively small to get an estimate, we approximate the nonadiabatic transition probability with its value in the Hermitian case where $\gamma = 0$. Unfortunately, even in this case no analytic expression for P_{nonad} is known [14]. Using a Landau-Zener type approximation, however, it can be estimated as

$$P_{nonad} \approx \exp\left(-2\text{Im} \int_{L/2}^{z_0} (E_+ - E_0) dz\right), \quad (15)$$

where z_0 denotes the position of the exceptional point nearest to the real z -axis [15].

In the Hermitian case exceptional points of all three levels (EP3) appear for $E_1 = E_2 = E_3 = 0$, corresponding to complex values of z given by the equation $v = \pm iw$.

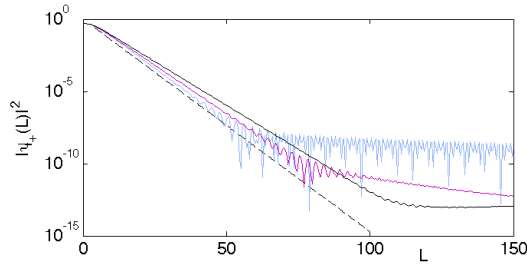


FIG. 4. Nonadiabatic transition probabilities as a function of L for $a = 5$ for different values of γ , $\gamma = 0$ (light blue), $\gamma = 0.1$ (magenta) and $\gamma = 0.25$ (black). For comparison also the estimation based on equation (17) is shown (black dashed line).

Using (4), we find that they are located at

$$z_n = L \left(\frac{1}{2} + i(1 + 2n) \frac{\pi}{4a} \right), \quad n \in \mathbb{Z}. \quad (16)$$

The EP3 with the smallest positive imaginary part of z is z_0 . From $E_0 = 0$ and $E_+(L/2 + i\xi) = \sqrt{v^2 + w^2} = \sqrt{2 \cos(2a\xi/L)}$, we can compute the integral in (15) as

$$\begin{aligned} P_{nonad} &\approx \exp \left(-2 \int_0^{\text{Im } z_0} \sqrt{2 \cos(2a\xi/L)} d\xi \right) \\ &= \exp \left(-\frac{2}{a\sqrt{\pi}} \Gamma^2\left(\frac{3}{4}\right) L \right). \end{aligned} \quad (17)$$

Substituting this expression into (14) yields an analytic estimate for the critical decay rate,

$$\gamma_{cr}^{LZ} = \frac{1}{L} \ln \left(\frac{\exp \left(\frac{2}{a\sqrt{\pi}} \Gamma^2\left(\frac{3}{4}\right) L \right)}{2} - 1 \right), \quad (18)$$

which for large values of L simplifies to

$$\gamma_{cr}^{LZ} \approx \frac{2}{a\sqrt{\pi}} \Gamma^2\left(\frac{3}{4}\right) - \frac{\ln(2)}{L}. \quad (19)$$

In Fig. 3 this value is shown as a white dashed line. It can be seen that it is a good estimate for the exact transition boundary. It fails, however, to describe a slow decrease of γ_{cr} for large values of L . This decrease is due to the fact that the Landau-Zener approximation for the

nonadiabatic corrections assumes a vanishing coupling at the beginning and the end of the time evolution, which is not the case in the STIRAP scheme investigated here.

The exact nonadiabatic transition probability can be obtained by a numerical integration. We show the result for three different values of γ in a semilogarithmic plot as a function of L in Fig. 4. The light blue line corresponds to the Hermitian case. For comparison, the Landau-Zener estimate (17) is also shown (dashed black line). The initially approximately exponential decrease of the transition probability with increasing L is well described by the Landau-Zener type expression. At a critical value of L , however, the transition probability begins to oscillate around an only slowly decreasing mean value. Similar behaviour is typically observed in STIRAP schemes [14]. The presence of decay leads to a smoothing of the oscillations, and an increase of the critical value of L . It can also be seen that the slope in the region of exponential decay is slightly decreased by the decay. This explains why the white lines in Fig. 3 slightly overestimate the critical value of γ . As expected the quality of the approximation is better for the case $a = 8$, where the boundary is located at smaller values of γ .

We can obtain an excellent approximation for the transfer boundary by using the numerical values of the nonadiabatic transition probability for a fixed value of $\gamma = 0.2$, close to the actual boundary, as an input in equation (14). This is demonstrated in Fig. 4, where the solid black lines depict the thus obtained result.

Conclusion. We have demonstrated that even a tiny decay rate can significantly influence the dynamical behaviour of a system, in particular in the context of adiabatic time evolutions. This is due to a competition between small nonadiabatic transition amplitudes and relative exponential growths of the decaying adiabatic eigenstates. In particular, we have shown for a STIRAP related scheme, which can be implemented straightforwardly using optical waveguides, that the adiabatic transfer behaviour breaks down at a sharp threshold for relatively small decay rates. The critical value of the decay rate has been estimated by simple analytical arguments.

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